An Analytical Study of a Periodic Multilayered Medium for Underwater Applications

Laetitia Roux (1,2), Christian Audoly (1), Christian Granger (3), Anne-Christine Hladky (3), Nicole Kessissoglou (2)

(1) Naval Group Research, Toulon-Ollioules, France
(2) School of Mechanical and Manufacturing Engineering, University of New South Wales, Sydney, Australia
(3) IEMN (Institut d'Électronique, de Microélectronique et de Nanotechnologies), ISEN Department, UMR 8520 CNRS (National Center for Scientific Research), Lille, France

ABSTRACT

Interest in the use of metamaterials has increased considerably in recent years as these materials exhibit unusual properties due to their internal structure. In this paper, a periodic multilayered medium formed by the alternation of stiff and soft materials is analytically examined, whereby the multilayered medium is modelled as a one-dimensional wave propagation problem. The transfer matrix method is used to determine the reflection and transmission coefficients on the incident and transmitted sides of the multilayered medium, respectively. The effective acoustic properties of the periodically layered medium in terms of the properties of an equivalent homogeneous medium are also derived using the transfer matrix method. These effective parameters, corresponding to the effective speed of sound and effective density, are compared to those obtained using a quasi-static approach. Dispersion curves for the periodically layered medium and results for the transmitted and reflected pressure are presented.

1 INTRODUCTION

Acoustic metamaterials are artificial materials which have been receiving growing interest in the last decade due to their capacity to exhibit unusual properties such as sub-wavelength band gaps, negative refraction and strong variations for the effective parameters. These unusual properties arise from the collective manifestation of the internal constituent units in the structure, such as resonant inclusions arranged in a host matrix (Ma and Sheng, 2016). A multilayered medium formed by the periodic alternation of stiff and soft layers is considered here. The material properties for this multilayered design can be derived in terms of effective parameters of an equivalent homogeneous medium, corresponding to the effective speed of sound and effective density. Effective parameters and dispersion relations for 1D longitudinal wave propagation in a periodic layered composite have been derived analytically (Rytov, 1956) and using approximation methods (Nemat-Nasser et al., 2011; Meresse, 2015). Metamaterials based on homogenisation of layered structures comprising two alternating materials in air have been designed as a broadband acoustic cloak (Torrent and Sanchez-Dehesa, 2011). For underwater applications, metamaterials have been employed as underwater acoustic barriers and as anechoic coatings applied to the outer hull of a submerged marine vessel. Leroy et al. (2015) numerically and experimentally examined a metamaterial comprising an elastic medium with bubble metascreens submerged in water, for broadband absorption in the ultrasonic frequency range. Sharma et al. (2016, 2017) developed a semi-analytical model based on homogenisation theory to examine the acoustic performance of a periodically voided soft elastic medium, without and with a steel backing, for underwater applications.

In this work, a periodic multilayered medium comprising alternating stiff and soft material layers is analytically examined for underwater applications. One-dimensional wave propagation through the multilayered pattern is derived in terms of the transfer matrix method. Reflection and transmission coefficients for the acoustic pressure on the incident and transmitted sides of the layered medium are obtained. The transfer matrix method is also implemented to compute the dispersion curves as well as the effective parameters corresponding to the effective speed of sound and effective density of an equivalent homogenised layer. Different configurations of the periodic unit are studied and the corresponding effective parameters are examined.

2 THEORETICAL METHODOLOGY

In this study, a multilayered medium comprising alternating layers of two materials whereby one layer is soft and the other layer is stiff is examined. The transfer matrix for one-dimensional sound propagation through the multilayered structure is initially derived, from which the reflection and transmission coefficients are obtained. The effective speed of sound and effective density of an equivalent homogeneous medium are then obtained from the constituting periodic unit comprising one stiff layer and one soft layer, also based on the transfer matrix method. Figure 1(a) represents a multilayered medium comprising three units of the symmetric periodic unit design shown in Figure 1(b) and immersed in water, while alternative periodic units (which are equivalent if the material was infinite) are shown by Figures 1(c), 1(d) and 1(e). For the multilayered medium examined in this work, the stiff material is aluminium and the soft material is silicone rubber.

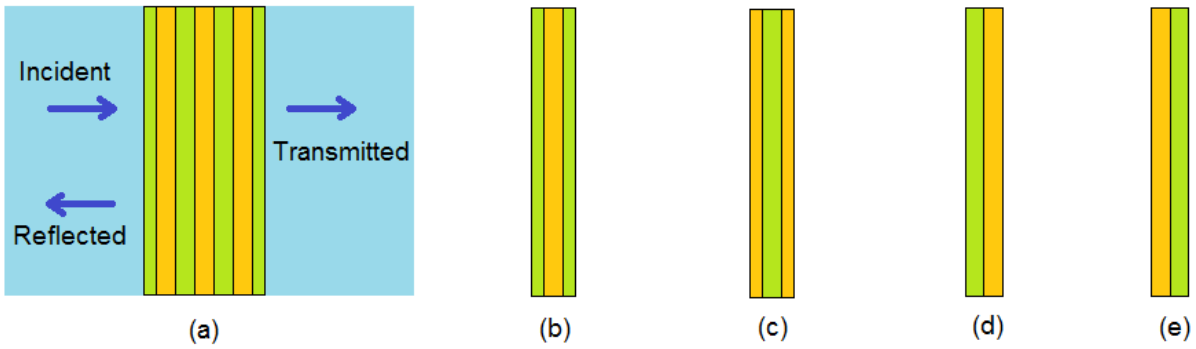

Figure 1: (a) A multilayered medium immersed in water and formed by three units of the periodic unit design given by (b). Alternative designs for the periodic unit are given by (c), (d) and (e). The silicone is represented by the yellow layers and aluminium is represented by the green layers.

2.1 Determination of the reflection and transmission coefficients using the transfer matrix method

The transfer matrix for 1D acoustic wave propagation through a layer relates the acoustic pressure and normal particle velocity of the fluid on either side of the medium and is given by (Munjal, 1987)

$$\mathbf{T} = \begin{bmatrix} \cos(kh) & \mathrm{i}\rho c \sin(kh) \\ \frac{\mathrm{i}.\sin(kh)}{\rho c} & \cos(kh) \end{bmatrix} \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary number, ρ is the density of the material, c is the speed of sound in the material, $k = \omega/c$ is the longitudinal wavenumber, ω is the radian frequency and h is the thickness of the layer.

For the case of a multilayered structure, the input acoustic pressure and normal particle velocity to a successive layer (layer $n+1$) corresponds to the output acoustic pressure and normal particle velocity of the previous layer (layer n). As such, the global transfer matrix of a structure comprising multiple layers is the product of the transfer matrices of each layer, that is

$$\mathbf{T}_\mathrm{g} = \prod_{n=N}^{n=1} \mathbf{T}_{\boldsymbol{n}} \tag{2}$$

where the subscript 'g' denotes global and $n = 1, \ldots, N$ corresponds to the number of layers. The acoustic pressure and normal particle velocity on the transmitted side of the multilayered medium due to an incident acoustic pressure can then be obtained using

$$\begin{bmatrix} p_\mathrm{t} \\ v_\mathrm{t} \end{bmatrix} = \mathbf{T}_\mathrm{g} \begin{bmatrix} p_\mathrm{inc} \\ v_\mathrm{inc} \end{bmatrix} \tag{3}$$

where the subscript 't' and 'inc' respectively correspond to transmission and incidence. The multilayered medium of global thickness h_g, represented by Figure 1(a), is immersed in water and subjected to a plane wave of unity amplitude at normal incidence. Once the transfer matrix of the multilayered medium is derived, its reflection and transmission coefficients are obtained as follows

$$\begin{bmatrix} Te^{-\mathrm{i}k_\mathrm{f}h_\mathrm{g}} \\ \frac{Te^{-\mathrm{i}k_\mathrm{f}h_\mathrm{g}}}{Z_\mathrm{f}} \end{bmatrix} = \mathbf{T}_\mathrm{g} \begin{bmatrix} 1+R \\ \frac{1-R}{Z_\mathrm{f}} \end{bmatrix} \tag{4}$$

where Z_f and k_f are respectively the impedance and wavenumber of the surrounding water, obtained by $Z_\mathrm{f} = \rho_\mathrm{f} c_\mathrm{f}$ and $k_\mathrm{f} = \omega/c_\mathrm{f}$.

2.2 Determination of effective parameters using the transfer matrix

In what follows, the periodic unit design represented by Figure 1(b), comprising a layer of aluminium followed by a layer of silicone and then another layer of aluminium, is now considered. The periodic unit is denoted by the subscript 'u'. A theoretical infinite repetition of this periodic unit can be described by an equivalent homogeneous medium with effective properties. These effective properties are derived from the transfer matrix $\mathbf{T}_\mathrm{u}$ of the periodic unit. The effective longitudinal wavenumber k_eff is obtained from the eigenvalues of the transfer matrix of the periodic unit $\mathbf{T}_\mathrm{u}$ using the following expression

$$\lambda = e^{\mathrm{i}k_\mathrm{eff}h_\mathrm{u}} \tag{5}$$

where h_u is the thickness of a single periodic unit, which in this case corresponds to the thickness of one layer of silicone plus the thickness of the two half layers of aluminium. Once the effective wavenumber is obtained by solving equation (5), the effective complex speed of sound is obtained using

$$c_\mathrm{eff} = \frac{\omega}{k_\mathrm{eff}} \tag{6}$$

Furthermore, using the real and imaginary parts of the effective speed of sound, the effective loss factor can be determined by

$$c_\mathrm{eff} = c' - \mathrm{i}c'' = c'(1 - \mathrm{i}\eta_\mathrm{eff}) \tag{7}$$

The effective density ρ_eff can be deduced from the transfer matrix of the periodic unit using either

$$\mathbf{T}_\mathrm{u}(1,2) = \mathrm{i}\rho_\mathrm{eff}c_\mathrm{eff}\sin(k_\mathrm{eff}h_\mathrm{u}) \quad \text{or} \quad \mathbf{T}_\mathrm{u}(2,1) = \mathrm{i}\frac{\sin(k_\mathrm{eff}h_\mathrm{u})}{\rho_\mathrm{eff}c_\mathrm{eff}}. \tag{8}$$

The effective impedance Z_eff is then obtained by

$$Z_\mathrm{eff} = \rho_\mathrm{eff}c_\mathrm{eff}. \tag{9}$$

The same method can be used to determine the effective parameters of other periodic patterns as shown in Figures 1(c), 1(d) and 1(e), whereby Figures 1(d) and 1(e) involve two layers of material instead of three.

2.3 Quasi-static effective parameters

At low frequencies, the thickness of the multilayered structure is small compared to the acoustic wavelength, resulting in low values for the longitudinal wavenumber. Using a quasi-static approach, the effective parameters of a periodic unit comprising two layers of material as represented by Figures 1(d) and 1(e), are obtained and compared with those effective properties obtained using the transfer matrix method at low frequencies. The quasi-static effective density is given by the average density for the various layers in the equivalent homogeneous medium as follows

$$\rho_\mathrm{eff}^\mathrm{QS} = \frac{\sum_{l=1}^{N} h_l\rho_l}{h_\mathrm{u}} \tag{10}$$

The effective wavenumber can be obtained using the dispersion relation as described in the previous section. The boundary condition at the interface between two consecutive layers is given by (Meresse, 2015)

$$\cos\left(k_\mathrm{eff}(h_1 + h_2)\right) = \cos(k_1h_1) \,.\, \cos(k_2h_2) - \frac{1}{2}\left(\gamma + \frac{1}{\gamma}\right) .\, \sin(k_1h_1) \,.\, \sin(k_2h_2) \tag{11}$$

where h_1, k_1, Z_1 and k_2, h_2, Z_2 are respectively the thickness, wavenumber and impedance of layers 1 and 2 of the periodic unit, and $\gamma = \frac{Z_1}{Z_2}$. At low frequencies, corresponding to low values of the reduced wavenumber, a polynomial approximation of equation (11) leads to

$$(k_\mathrm{eff}^\mathrm{QS})^2(h_1 + h_2)^2 \cong (k_1h_1)^2 + (k_2h_2)^2 + \left(\gamma + \frac{1}{\gamma}\right) k_1h_1k_2h_2 \tag{12}$$

Using $k = \frac{\omega}{c}$, an explicit expression for the effective sound speed is obtained as follows

$$c_{\text{eff}}^{\text{QS}} \cong \frac{h_1+h_2}{\sqrt{(\frac{h_1}{c_1})^2+(\frac{h_2}{c_2})^2+\left(\gamma+\frac{1}{\gamma}\right)\frac{h_1 h_2}{c_1 c_2}}}. \tag{13}$$

The quasi-static effective impedance Z_{eff} is then obtained by

$$Z_{\text{eff}}^{\text{QS}} = \rho_{\text{eff}}^{\text{QS}} c_{\text{eff}}^{\text{QS}}. \tag{14}$$

3 RESULTS

Results for a multilayered medium formed by periodic layers of silicone and aluminium are presented. The complex speed of sound and density for silicone are given by $c_{\text{silicone}} = 1000(1 - \mathrm{i}\eta_{\text{silicone}})\ \mathrm{m/s}$ where η_{silicone} is the loss factor and $\rho_{\text{silicone}} = 1250\ \mathrm{kg/m^3}$. Similarly, the speed of sound and density for aluminium are given by $c_{\text{aluminium}} = 6200\ \mathrm{m/s}$ and $\rho_{\text{aluminium}} = 2700\ \mathrm{kg/m^3}$. The speed of sound and density in water on the incidence and transmission sides of the multilayered medium are $c_{\text{f}} = 1500\ \mathrm{m/s}$ and $\rho_{\text{f}} = 1000\ \mathrm{kg/m^3}$.

3.1 Dispersion curves

Figure 2 presents the dispersion curves of the equivalent homogeneous medium representing an infinite periodic medium using the expressions for the effective parameters derived in section 2.2. The periodic unit corresponds to Figure 1(b) and consists of a layer of aluminium of 5 mm thickness followed by a layer of silicone of 30 mm thickness and then another layer of aluminium of 5 mm thickness. In the generation of the effective parameters for the equivalent homogeneous medium, different values of the silicone loss factor η_{silicone} were examined. Dispersion curves in terms of the dimensionless (reduced) wavenumber denoted by $k_{\text{reduced}} = k_{\text{eff}} h_u$ are presented for a frequency range up to 40 kHz. For the frequency range considered, two complete band gaps appear. Assuming zero damping, distinct frequency ranges corresponding to stop bands occur where the real part of the effective reduced wavenumber is equal to 0 or π. This is attributed to the fact that at the interface between the layers of aluminium and silicone, an incoming wave is transferred into a reflected wave which constructively interferes with the incoming wave. Figure 2 shows that as the loss factor of the silicone is increased, the band gap vanishes and the imaginary part of the wavenumber is not zero outside the initial stop bands. This finding was recently reported by Meresse et al. (2015) who showed that the increase of damping reduced the clear bound of band gaps in periodic structures.

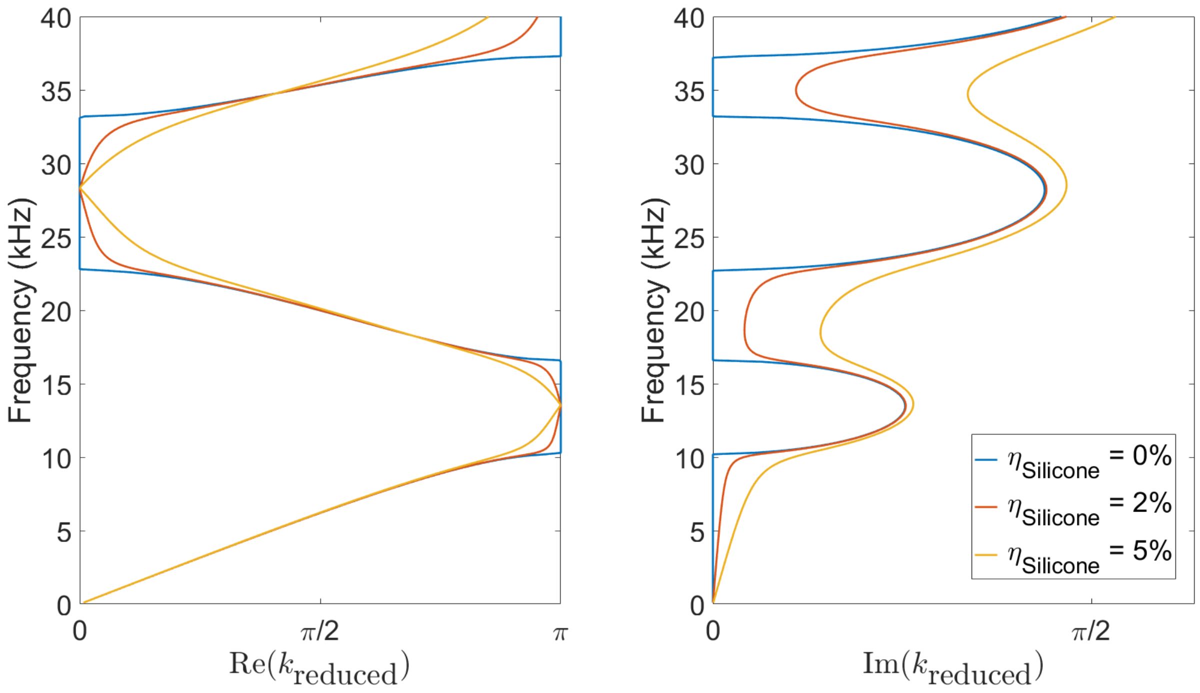

Figure 2: Dispersion curves of the equivalent homogeneous medium for different values of the loss factor for silicone.

3.2 Effective parameters

Figure 3 presents the real part of the effective sound speed c_{eff}, the effective loss factor η_{eff}, the magnitude of the complex effective density ρ_{eff} and the corresponding effective impedance Z_{eff} of the equivalent homogeneous medium for the same periodic unit represented by Figure 1(b), for a silicone loss factor of $\eta_{\mathrm{silicone}} = 2\%$. The blue shaded areas represent the band gaps created by the equivalent homogeneous medium for the case of zero damping for the silicone layer, corresponding to the blue curve in Figure 2. Within the band gaps in Figures 2 and 3, the effective parameters exhibit distinct behaviour which is summarised as follows.

In Figure 2, the imaginary part of the wavenumber (which represents the wave attenuation per unit distance) reaches a local maximum, corresponding to greater attenuation of acoustic waves in the stop bands than in the pass bands. Furthermore, the local maxima increase with frequency, resulting in greater attenuation of the acoustic waves with increasing frequency. In Figure 3, it is clear that the real part of the effective speed of sound is lower within the second stop band than for the surrounding pass bands. This implies that the propagation of the waves in the effective medium is slow within the stop band. In Figure 3, the effective loss factor reaches a local maximum centred in the stop bands and becomes a very high value in the second stop band (although not shown in the figure for a clearer view of the first peak), implying that the effective medium is significantly intrinsically damped. This indicates that the wave amplitude rapidly decreases such that the propagating wave becomes an evanescent wave, thus characterising the band gap. The effective impedance exhibits similar characteristics for each band gap, having a local minimum at the first branch of the band gap and then growing to a local maximum at the end of the band gap. Consequently, variations of the effective impedance determine the bounds of the band gaps which may not be easy to distinguish in the dispersion curves when the loss factor for the silicone damping is not zero. In the graph for the effective impedance, the water impedance is represented by the horizontal dashed-dotted line.

The dispersion curves and effective parameters for the four configurations of the periodic units shown in Figures 1(b) to 1(e) were compared, whereby for each case, the thickness of the equivalent homogeneous layer h_{u} is the same. Further, the overall thickness of silicone for each case is the same, and similarly, the overall thickness of aluminium for each case is also the same. As described previously, the first case of the equivalent homogeneous medium as shown in Figure 1(b) corresponds to a 5 mm thick layer of aluminium is followed by a 30 mm thick layer of silicone and then another 5 mm thick layer of aluminium. In the second case shown in Figure 1(c), the layers of aluminium and silicone are reversed such that the periodic unit consists of a 15 mm thick layer of silicone followed by a 10 mm thick layer of aluminium and then followed by another 15 mm thick layer of silicone. The third case shown in Figure 1(d) corresponds to a layer of aluminium of 10 mm thickness followed by a layer of silicone of 30 mm thickness. The fourth case shown in Figure 1(e) corresponds to one layer of silicone of thickness 30 mm, followed by one layer of aluminium of 10 mm thickness. Note that the first two cases shown by Figures 1(b) and 1(c) are symmetric configurations whereas the last two cases shown by Figures 1(d) and 1(e) are non-symmetric cases in which the two layers are reversed.

For all four cases, dispersion curves for the non-dimensional wavenumber are the same. Furthermore, the effective speed of sound is deduced from the effective wavenumber and as such it is also not affected by the choice of the periodic unit design.

The effective density of the equivalent homogeneous material was calculated using either $\mathbf{T}_{\mathrm{u}}(1{,}2)$ or $\mathbf{T}_{\mathrm{u}}(2{,}1)$ of the global transfer matrix for each design, as given by Eq. (8). For a symmetric periodic unit represented by either Figure 1(b) or 1(c), it was observed that the effective density is the same using either $\mathbf{T}_{\mathrm{u}}(1{,}2)$ or $\mathbf{T}_{\mathrm{u}}(2{,}1)$ of the global transfer matrix. However, for a non-symmetric periodic unit represented by Figure 1(d) or 1(e), the effective density obtained using $\mathbf{T}_{\mathrm{u}}(1{,}2)$ or $\mathbf{T}_{\mathrm{u}}(2{,}1)$ of the global transfer matrix was not the same (although the results are not shown here). As such, the effective density depends on the design of the periodic unit, which leads to the conclusion that the transfer matrix method is not suitable to determine the effective density for the case of a non-symmetric periodic unit.

In Figure 3, the horizontal dashed lines correspond to the quasi-static values of the effective parameters and are the same for all four designs of the periodic unit represented by Figures 1(b) to 1(e), thus validating the effective parameters of the equivalent homogeneous medium at low frequencies. It can be observed that at low frequencies, the quasi-static effective impedance is almost equal to the impedance of water.

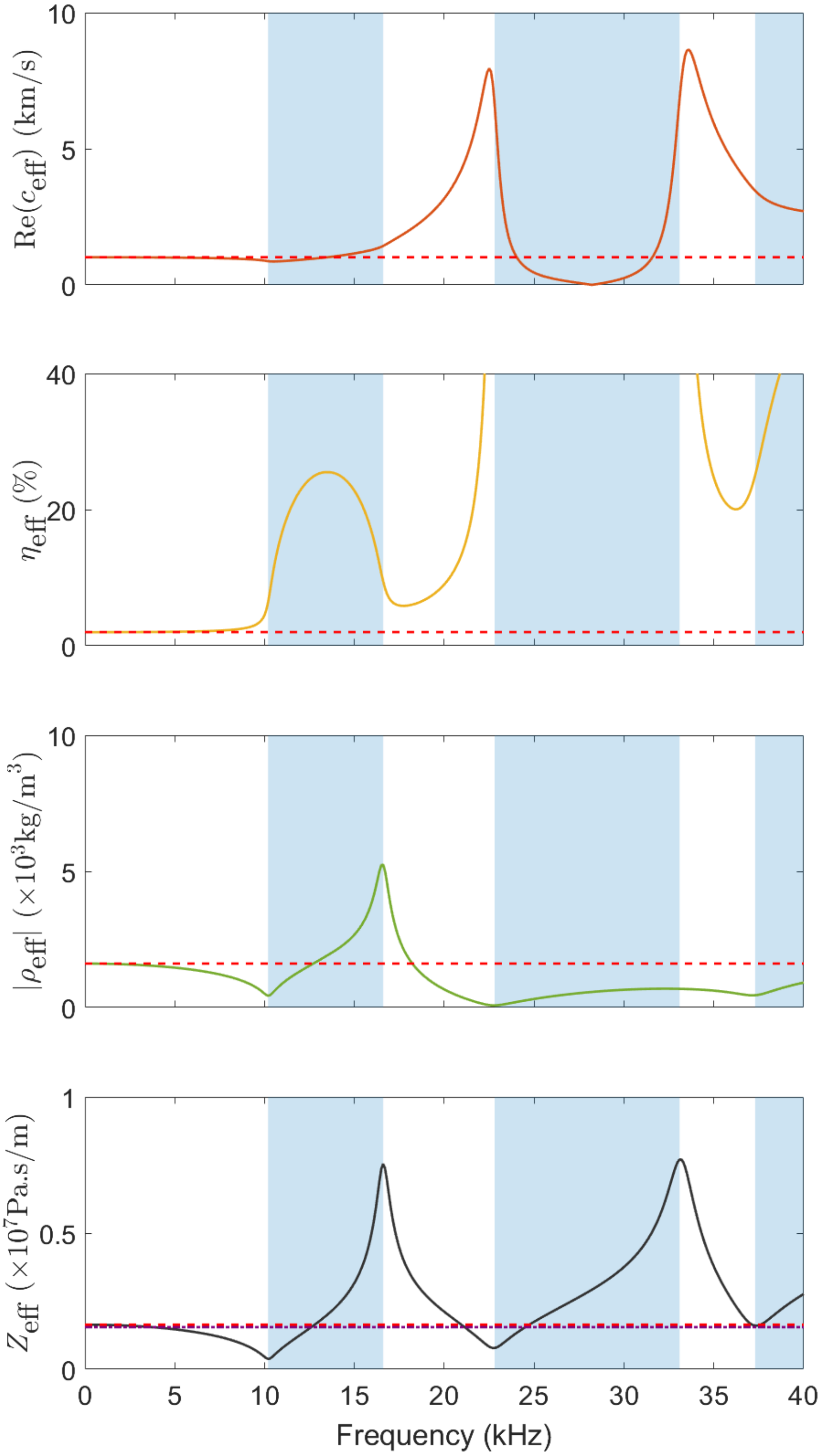

Figure 3: Effective parameters for the symmetric periodic unit represented by Figure 1(b), corresponding to the real part of the effective sound speed, the effective loss factor, the modulus of the effective density and the effective impedance. The blue shaded areas represent the band gaps in the dispersion results in Figure 1 for zero loss factor of the silicone layer (represented by the blue curve results in Figure 1). The horizontal dashed lines correspond to the quasi-static values of the effective parameters. In the graph of the effective impedance, the horizontal dashed-dotted line represents the water impedance.

3.3 Transmission and reflection coefficients

The transmission and reflection coefficients are computed using the effective properties obtained from the two unit designs represented by Figures 1(b) and 1(c), for which both designs are symmetric but the soft and stiff layers are reversed. The transmission and reflection coefficients of the multilayered media in water have also been obtained analytically using the methodology described in section 2.1. The coefficients have been calculated for multilayered media comprising one, three and ten units of a periodic unit design. The effective density was calculated using $\mathbf{T}_{\mathrm{u}}(2,1)$ of the transfer matrix for each design. With regards to Figures 4 and 5, the following observations are made.

- For the symmetric unit designs represented by Figures 1(b) and 1(c), the transmission and reflection coefficients obtained analytically for the multilayered media are identical to those obtained for an equivalent homogeneous medium.
- As the number of periodic units increase, the presence of the band gaps becomes apparent, whereby band gaps associated with low transmission coefficient values corresponds to high reflection coefficients.
- The results for the two designs become similar to each other as the number of periodic units increase. This implies that for the choice of aluminium and silicone to respectively represent the stiff and soft layers, the coefficients are less sensitive to the material on the incidence and transmission sides of the multilayered medium.

4 SUMMARY

In this work, a multilayered medium comprising alternating layers of a soft material corresponding to silicone rubber and a stiff material corresponding to aluminium is examined. Four different configurations of a periodic unit are studied, whereby two designs correspond to symmetric periodic units and the other two designs correspond to non-symmetric periodic units in which the soft and stiff layers are reversed. For each periodic unit configuration, the effective parameters and reflection and transmission coefficients are obtained using the transfer matrix method. For all four periodic unit designs, dispersion curves for the effective wavenumber and the effective speed of sound were the same. However, differences in the effective density between the symmetric and non-symmetric periodic units were found to occur. For the symmetric periodic designs, results for the reflection and transmission coefficients of the effective homogeneous medium were identical to those obtained from the analytical model. Furthermore, results for the two symmetric designs for which the soft and stiff layers were reversed were observed to converge with increasing number of periodic units.

REFERENCES

Leroy, V., Strybulevych, A., Lanoy, M., Lemoult, F. and Tourin, A. 2015. ‘Superabsorption of acoustic waves with bubble metascreens’. *Physical Review B* 91 (2): 020301.

Ma, G. and Sheng P. 2016. ‘Acoustic metamaterials: From local resonances to broad horizons’. *Science Advances.* 2, e1501595.

Meresse, P. 2015. ‘Matériaux absorbants à structure périodique et inclusions résonantes pour l’acoustique sous-marine’. PhD Thesis. Ecole Doctorale Sciences pour l’Ingénieur, Lille, France

Meresse, P., Renou, Y., Vasseur, J., Audoly, C. and Hladky, A.C. 2015. Propagation number in periodic structures considering losses, J. Phys. Conf. Ser. 581 (1): 012012.

Munjal, M.L. 1987. ‘Acoustics of ducts and mufflers with application to exhaust and ventilation system design’. *John Wiley & Sons.*

Nemat-Nasser, S., Willis, J. R., Srivastava, A. and Amirkhizi, A.V. 2011. ‘Homogenization of periodic elastic composites and locally resonant sonic materials’. *Physical Review B* 83 (10): 104103.

Rytov, S. 1956. ‘Acoustical properties of a thinly laminated medium’. *Soviet Physics Acoustics* 2: 68-80.

Sharma, G.S., Skvortsov, A., MacGillivray, I., and Kessissoglou, N. 2017. ‘Acoustic performance of gratings of cylindrical voids in a soft elastic medium with a steel backing’. *The Journal of the Acoustical Society of America* 141 (6): 4694-4704.

Sharma, G.S., Skvortsov, A., MacGillivray, I. and Kessissoglou, N. 2017. ‘Sound transmission through a periodically voided soft elastic medium submerged in water’. *Wave Motion* 70: 101-112.

Torrent, D. and Sánchez-Dehesa, J. 2011. ‘Broadband acoustic cloaks based on the homogenization of layered materials’. *Wave Motion* 48: 497-504.

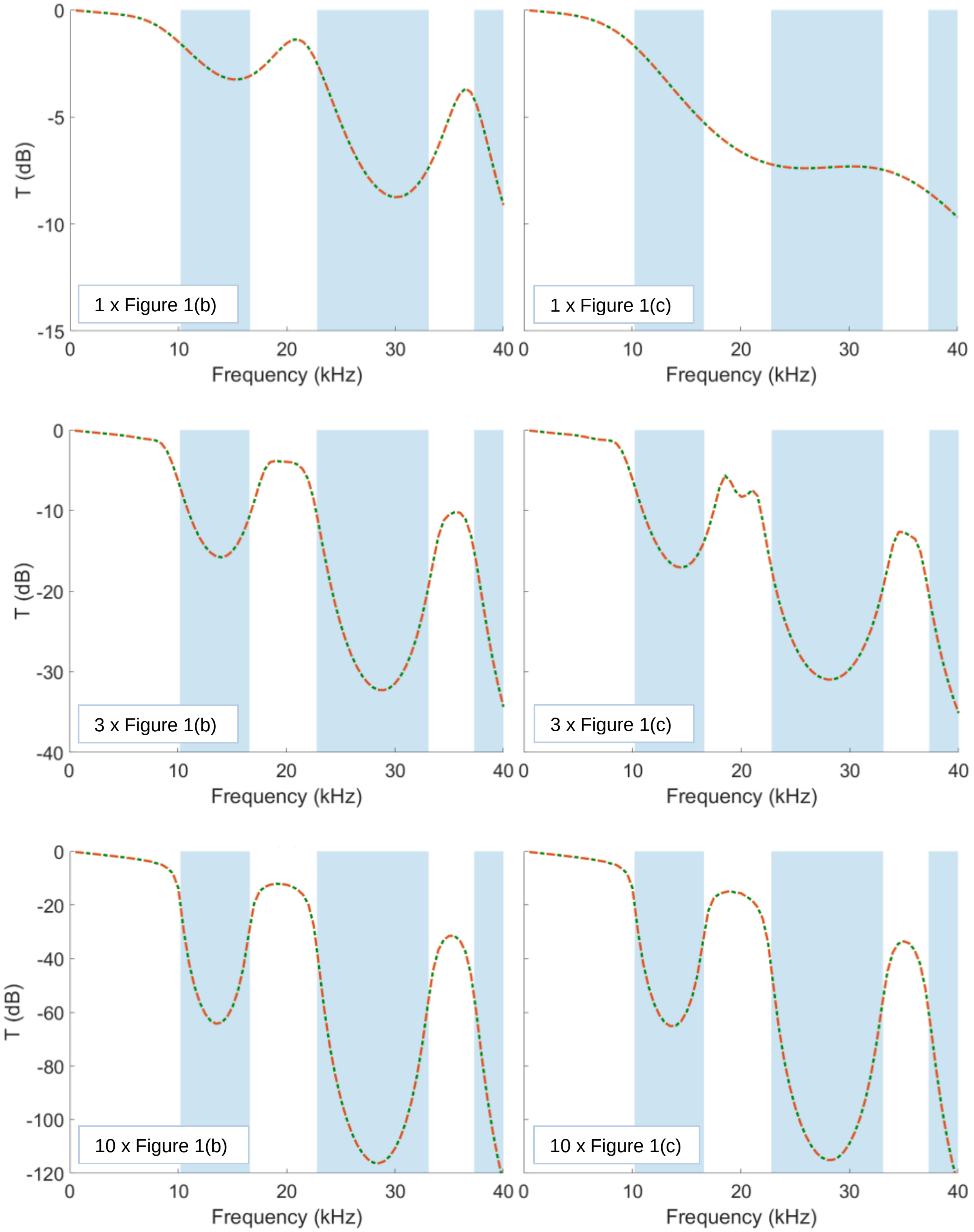

Figure 4: Comparison of transmission coefficients obtained analytically for a multilayered medium (green dotted line) and for the equivalent homogeneous medium (orange dashed line) comprising 1 unit (top), 3 units (middle) and 10 units (bottom) of the symmetric unit designs represented by Figure 1(b) (left column) and Figure 1(c) (right column).

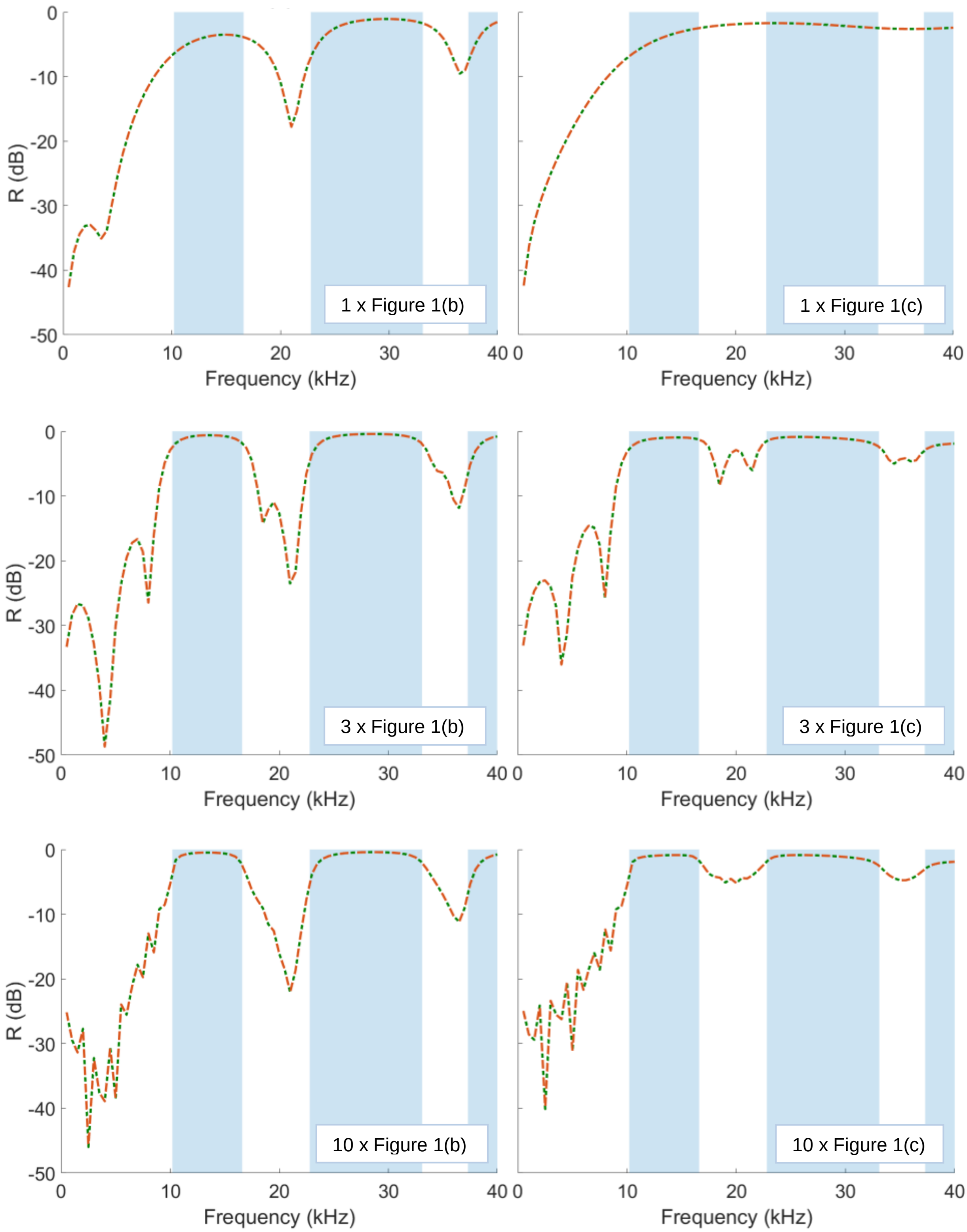

Figure 5: Comparison of reflection coefficients obtained analytically for a multilayered medium (green dotted line) and for the equivalent homogeneous medium (orange dashed line) comprising 1 unit (top), 3 units (middle) and 10 units (bottom) of the symmetric unit designs represented by Figure 1(b) (left column) and Figure 1(c) (right column).