ON THE COMPLETENESS AND ORTHOGONALITY OF THE ACOUSTIC MODES OF AN OPEN CAVITY

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Abstract

The modal representation of closed acoustical systems such as rooms and enclosures is well known and is commonly used to find the forced response at a given frequency as the superposition of the modes. In open systems such as parallel noise barriers, open enclosures or reactive type duct mufflers, the modal representation may be incomplete due to radiation losses encompassed by the imaginary part of the eigensolution. This has been declared as an open problem in the literature, with sound field predictions being found accurate only at the resonant frequencies. In the present study, the completeness and orthogonality of the quasinormal modes is investigated using the example of a two-dimensional open cavity system.

1. Introduction

The normal modes of an acoustic system are useful in the understanding of concepts such as nodes and anti-nodes, and may be used in designing noise control solutions. Furthermore, the forced response as calculated from the eigensolution may be more computationally efficient than finite element methods [1, 2]. An open system is one which loses energy, having a complex eigensolution consisting of so called 'quasinormal' modes. In this paper we are concerned with open systems that have an infinite physical domain where sound radiates to infinity, in contrast to systems with a closed domain and absorptive surfaces described by an impedance boundary condition which also has complex eigensolutions.

Not much literature exists on calculating the forced response using the modal description of sound radiated from open systems, despite their significant practical importance. Recently, Yang [3] attempted to examine the mechanisms of a ‘wave-trapping barrier’ (a noise barrier that has surfaces geometrically designed to reflect waves downwards) by solving the eigenproblem numerically using a perfectly matched layer (PML). Upon applying the same equation for the forced response that is commonly used in room acoustics [4] (Eq. (1)) over the numerical domain it was found that the transmission loss values calculated were only reliable at frequencies close to the eigenvalues and inaccurate elsewhere. Thus the problem of how to calculate the transmission loss correctly was proposed as an open question.

$$p(x,y) = -j\omega Q \sum_I \frac{\phi_I(x,y)\, \phi_I^*(x_0,y_0)}{(k^2-k_I^2)\int_\Omega \phi_I(x,y)\phi_I^*(x,y)\, d\Omega} \tag{1}$$

Here we define $j=\sqrt{-1}$, $\phi_I(x,y)$ as the mode shape, Q as the volume velocity, ω as the frequency of excitation, $k = \omega/c$ as the wavenumber for sound speed c, k_I as the mode wavenumber related to the eigenvalue, and Ω as the physical domain.

The derivation of Eq. (1) has made the assumptions of completeness and orthogonality, which may not hold true for problems involving an open system that radiates energy to infinity. For example, in the study of ocean waveguides completeness is violated when the seabed is treated as an infinite acoustic medium. In these problems one will obtain a mixed spectrum of eigenvalues consisting of a discrete and continuous part. The continuous part may be evaluated using a complex contour integral, however is often neglected in practical problems in ocean acoustics when one can assume a sufficiently large distance from the source [1].

In one dimensional open systems, a mathematical proof exists for the completeness of any system provided that it contains a weak discontinuity between the system and the bath, and that it satisfies the no-tail condition of zero reflections coming back from infinity [5]. This work was targeted at understanding optical systems where a material discontinuity is the norm. The result has been applied to one dimensional acoustic problems involving resistive boundaries [6] and the analysis of the transmission loss of duct expansion chambers [7]. In acoustical problems of dimension greater than one, the discontinuity and no-tail conditions require further analysis.

As for the calculations of the quasinormal modes themselves, there exists both analytical and numerical approaches in the acoustics literature. Tam [8] formulated an analytical solution for the problem of a two-dimensional open cavity in an infinite baffle which can be solved with the aid of a root finding method. Koch [9] used a numerical eigensolver which truncated the discretised infinite domain using a perfectly matched layer (PML) to solve the same problem as Tam. Agreement was reasonable at the first few modes except for those with strong coupling to the semi-infinite half space (large imaginary part). The theoretical predictions have been validated experimentally [10]. The PML used in the calculation of leaky modes in a waveguide has an unusual requirement in that it should be placed as close as possible to the discontinuity as possible, otherwise spurious modes and a perturbed result for the leaky modes will occur due to the nature of the complex coordinate transform [2].

An investigation into the response as the sum of modes has been undertaken for efficiently calculating the propagation of sound in street canyons, modelled using a 3D extension to the open cavity problem [11], however whilst the lack of orthogonality was highlighted, the issue of completeness and the exterior field was not directly addressed.

The objective of this paper is therefore to investigate the completeness and orthogonality relations of an open acoustical system in two or more dimensions and to provide a formulation for the forced response of open systems. In particular, a source placed inside the cavity is representative of the noise barrier problem. First, the representation of an open system using a continuous set of modes is discussed. The two-dimensional open cavity problem is investigated using a large set of modes generated using Tam's analytical solution so that the lossy modes may be calculated in a reliable manner, and new features in the oscillation of the radiation loss with cavity dimension are observed. A finite element analysis (FEA) solution is provided as a reference solution for the forced response, and by comparing solutions, the completeness and orthogonality of the system may be revealed.

2. Eigensolution of the infinite domain

Consider the simple problem of a point-line source placed in a 2D half-space (Figure 1a). The geometry will yield a continuous spectrum of eigenvalues ($k \in \mathbb{R}$) satisfying the homogeneous wave equation. On the other hand, let us also consider the same source next to the wall, but now enclosed by a rigid box (Figure 1b).

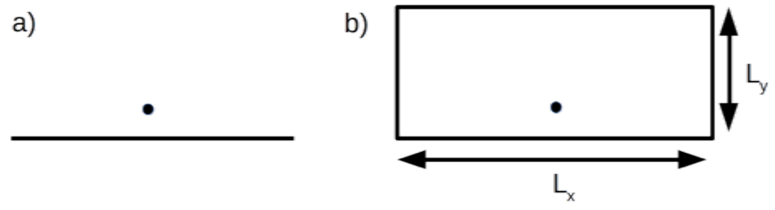

Figure 1. Point-line source placed in: a) infinite half-space, b) enclosed system

The solution for the forced response of the half-space problem may be calculated using Eq. (2) for the enclosed system under the limit of $L_x, L_y \to \infty$ with some small damping added to the eigenvalue $\hat{k}_I$ of the enclosed system: $k_I = \hat{k}_I + j\xi$, where $\xi \to 0^+$. The damping ensures that the reflected waves are of negligible amplitude, and ξ is small enough such that it does not noticeably effect the solution in the region of interest. This allows us to use a discrete approximation to the continuous eigenvalue spectrum, with the forced response shown to match the analytical solution, although it is much more tedious to calculate.

$$p(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} D_{mn} \cos\left(\frac{m\pi x}{L_x}\right)\cos\left(\frac{n\pi y}{L_y}\right). \tag{2}$$

The same solution may be arrived at by evaluating the large arc contour path of the inverse Fourier transform and then taking a forward transform of the result; the continuous modes are not poles. The condition for completeness of the discrete modes is that the Green's function of the system must vanish as $|\omega| \to \infty$ in the lower half plane so that the response is described fully by the poles [5]. This condition is not met for the half-space problem.

2.1 Eigensolution of an open cavity

A more interesting question is now to consider the two coupled spaces (the cavity Ω_1 and the exterior space Ω_2) which form the open cavity problem (Figure 2).

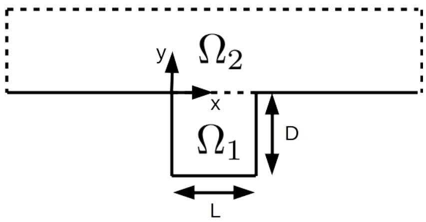

Figure 2. Open cavity system consisting of domains Ω_1 (cavity space) and Ω_2 (external space)

One may calculate discrete eigenvalues as follows. Using Tam's approach we place a point line source in Ω_2 noting that the modes will be independent of the source location. The system must satisfy the forced wave equation:

$$[\nabla^2 + k^2]\, p(x,y) = -j\omega Q\delta(x - x_0)\delta(y - y_0). \tag{3}$$

The solution of Eq. (3) inside Ω_1 which satisfies the boundary conditions at the three rigid walls and joined conditions at the opening is sought by the expansion of admissible functions:

$$\phi(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{n\,\pi x}{L}\right)^2}\,(y + D)\right). \tag{4}$$

Placing the source in Ω_2 allows an expression for the solution to (3) in Ω_2 and it is possible to join the two equations using the continuity conditions at the interface, leading to a matrix equation [8]:

$$[\Delta]\{A\} = \{b\}, \tag{5}$$

where the definitions of $[\Delta]$ and $\{b\}$ are functions of k, and are given in the appendix. For the I^{th} eigenvalue k_I, the coefficients $\{A\}$ for the that describe the field in Ω_1 can be calculated by applying

Cramer's rule. The eigenvalues for the poles in the response (zeros of $\det[\Delta]$) are solved numerically using Newtons method, using a set of starting guesses at a deep cavity with $D \gg L$ which may be approximated by a closed system with a pressure release boundary condition. Stepping the solution in small increments to the desired D/L cavity is then possible by using the last eigensolution as an initial guess for k_I.

The poles are not the complete set of eigensolutions to the homogeneous equation (Eq. (3) with $Q = 0$). A continuous set of solutions $k \in \mathbb{R}$ for $\{A\} = 0$. As the coefficients of the admissible functions have been set to zero, the solution will have an amplitude of zero inside the cavity and therefore the modal coefficients $\{D\}$ of the forced response will also be zero. The exterior region Ω_2 will indeed, on the other hand, have a continuous spectrum of non-zero eigenmodes.

Truncating the domain Ω_2 with rigid walls of size $L_x, L_y \to \infty$ will also tend towards a continuous solution of modes. The Ω_2 dominated modes of the finite system have a small amplitude inside the cavity, which is larger near the domain interface; however, as the dimension increases, the pressure in Ω_1 tends to zero (Figure 3). Therefore, the continuous modes do not affect the response $p: (x, y) \in \Omega_1 \cup \Omega_2$ from a source placed inside the cavity Ω_1.

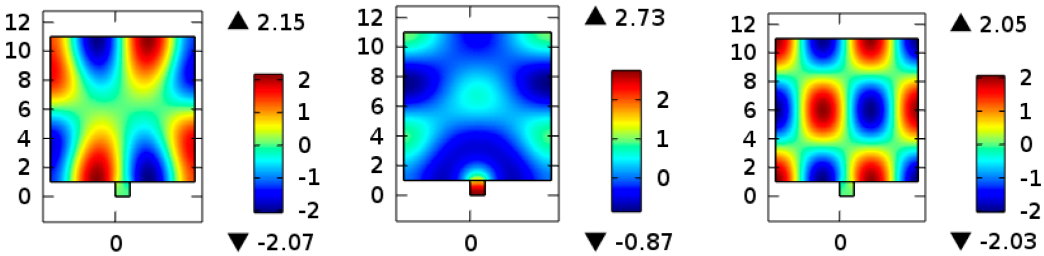

Figure 3. Modes of closed coupled spaces, left, right: Ω_2 dominated, center: Ω_1 dominated

Furthermore, it is possible to show that with a source placed inside $\Omega_1$that the response tends to zero with $\omega \to \infty$. Thus, there the continuous spectrum has no contribution to the response and the discrete modes form a complete set. In Eq. (5) the LHS behaves like $A_n\omega^{\frac{5}{2}}$ and the RHS like $e^{-\omega}$ for $\omega \to \infty$ for the source placed in Ω_1, so all A_n must go to zero. On the other hand, a source placed in Ω_2 has a RHS of $e^{+\omega}$.

3. The forced response of an open cavity as the sum of modes

When considering the forced problem, we wish to make use of the completeness relationship inside the cavity to substitute the modes back into Eq. (3) for mathematical manipulation:

$$p(x, y) = \sum_{I=0}^{\infty} D_I \phi_I(x, y), \quad (x, y) \in \Omega_1. \tag{6}$$

If completeness holds, then by substituting (6) in (3), multiplying by a conjugate mode shape $\phi_{I'}^*(x, y)$, and integrating both sides, one obtains

$$\iint_{\Omega_1} \sum\nolimits_I (k^2 - k_I^2) D_n \phi_I(\boldsymbol{x}) \phi_{I'}^* \, dx\, dy = -j\omega Q \iint_{\Omega_1} \delta(\boldsymbol{x} - \boldsymbol{x}_0) \phi_{I'}^*(\boldsymbol{x}) \, dx\, dy, \tag{7}$$

and we are able to examine the orthogonality of the of the modes by analysing

$$\textstyle\iint_{\Omega_1} \phi_I(x, y) \phi_{I'}^*(x, y) \, dx\, dy. \tag{8}$$

For a problem involving a closed boundary described by an impedance, the problem of solving Eq. (3) with $Q = 0$ is characterized as a regular Sturm-Liouville problem which guarantees orthogonality [12], and Eq. (7) in (6) reduces to (1). However, this is not the case since the domain is open and infinite, and we are effectively evaluating only a subdomain of the mode shape.

Without assuming anything about the orthogonality, we can include the cross terms in the following matrix equation

$$\left[\, \Lambda_{I',I} \right] \{(k^2 - k_I^2) D_I\} = \, -j\omega Q \, \{\phi_{I'}^*(x_0, y_0)\}, \tag{9}$$

where

$$\Lambda_{I,I'} = \iint_{\Omega_1} \phi_I(x,y) \phi_{I'}^*(x,y) \, dx \, dy. \tag{10}$$

Substituting the solution for D_I into (6) will yield the forced response assuming that completeness has been satisfied. If the mode shape inside the cavity is zero then D_I will be zero and the mode will not contribute to the forced response outside the cavity.

4. Numerical solution for open cavity system

4.1 Eigensolution

For the purpose of solving the first 5×5 eigenvalues, a $[\Delta]$ matrix size of $N' = 20$ to implement the truncated sum appears to offer reasonable convergence and match well with the first few values which Tam provided. Tables 1 and 2 tabulate the convergence for a cavity of dimension $D/L = 1$.

The solution is plotted for various cavity dimensions D/L in Figure 4. Here the modes are labelled (m, n) with m corresponding to the depth mode and n to the width mode. The numbering is determined by continuity of the solution at large D/L. As expected, the amplitude of the imaginary part of the eigenvalue increases for shorter cavities – indicating a larger radiation loss. For long deep cavities, the imaginary part approaches zero since most of the energy becomes stored in the cavity. An interesting characteristic of the solution is the oscillatory behaviour of the imaginary part for the modes with $n > 1$. Presumably this is a feature of the radiation efficiency of the cavity given the complex distribution of sound pressure at the opening which depends on the ratio of the sound wavelength to the aperture length L.

Table 1. Normalised eigenvalue calculations: $\omega_I L/c$

m	n	$N' = 5$	$N' = 10$	$N' = 20$
0	0	0.955 - 0.301i	0.956 - 0.302i	0.956 - 0.303i
0	1	3.379 - 0.055i	3.380 - 0.056i	3.380 - 0.056i
0	2	6.423 - 0.030i	6.423 - 0.030i	6.423 - 0.031i
1	0	3.561 - 0.954i	3.567 - 0.962i	3.569 - 0.967i
1	1	4.987 - 0.405i	4.997 - 0.412i	4.999 - 0.414i
1	2	7.446 - 0.223i	7.451 - 0.226i	7.453 - 0.228i
2	0	6.260 - 1.870i	6.290 - 1.887i	6.302 - 1.899i
2	1	7.377 - 0.912i	7.418 - 0.931i	7.425 - 0.938i
2	2	9.333 - 0.615i	9.352 - 0.629i	9.359 - 0.637i

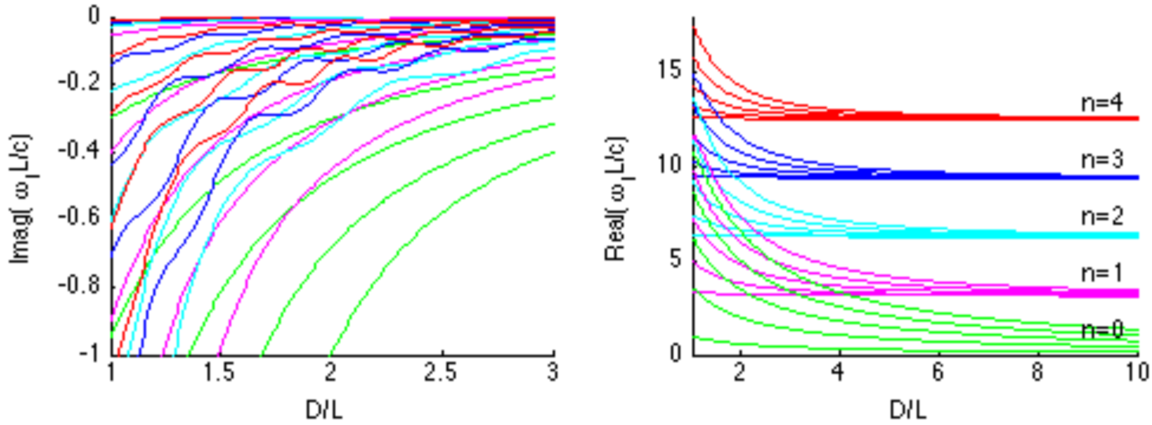

Figure 4. Eigenvalues ($\omega_I L/c$) for the open cavity problem: imaginary part (left), real part (right). Modes grouped in colour by width mode number n. First 5×5 modes are shown

4.2 FEA reference solution

In the following analysis we consider a cavity of $D/L = 1$ with a point-line source placed at $(L/10, -9D/10)$ for the purpose of comparing the forced response to the sum of modes. The forced response is presented in terms of physical parameter values rather than non-dimensional. We consider a 1.0×1.0 m cavity with an excitation frequency of 2000 rad/s. The material parameters are taken as a sound speed of 343 m/s and a density of 1.21 kg/m^3. This corresponds to a normalised frequency of $\omega L/c = 5.83$ and was chosen so as not to coincide with the resonant frequencies.

The FEA solution to the forced equation was implemented with the source represented by a smooth Gaussian shape implemented over a radius of 0.05 m and satisfying an integral of $Q = 1$. Convergence was observed against mesh fineness, tolerance, baffle length, PML thickness and the sharpness of the source distribution. An example of one of the coarser meshes used in the convergence study is given in Figure 5 alongside the final result. The PML region is shown to absorb the waves with no noticeable reflections.

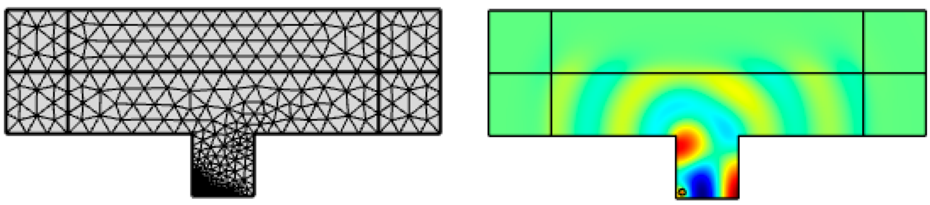

Figure 5. Example of mesh used for FEA (left), and converged solution at 2000 rad/s (right)

4.3 Agreement of forced response with FEA

We now compare the forced response as calculated using the matrix approach of Eq. (9) which includes the contribution of the cross terms to Eq. (1) which assumed orthogonality (i.e. non-diagonal elements of $[\Lambda]$ set to zero) with respect to the FEA solution. The first 25 modes are used in the comparison, corresponding to $m, n = \{0,1,2,3,4\}$, and the convergence with regards to the matrix size at three receiver locations are displayed in Tables 2 and 3. the domain response and the difference to the FEA solution are given in Figure 5, and Figure 7 shows the contribution of each mode to the forced response as measured by the magnitude of the coefficient D_I.

Table 2. Convergence of modal solution (matrix approach), first 25 modes

N'	R1 (0.9,-0.15)	R2 (0.9,-0.1)	R3 (0.9,-0.05)
5	0.1767 - 0.0999i	0.1959 - 0.0918i	0.2064 - 0.0745i
10	0.1386 - 0.1351i	0.1657 - 0.1199i	0.1837 - 0.0944i
20	0.1347 - 0.1390i	0.1633 - 0.1229i	0.1827 - 0.0963i
25	0.1344 - 0.1392i	0.1632 - 0.1231i	0.1827 - 0.0964i

Table 3. Convergence of modal solution (orthogonal approach), first 25 modes

N'	R1 (0.9,-0.15)	R2 (0.9,-0.1)	R3 (0.9,-0.05)
5	0.1672 - 0.0529i	0.1664 - 0.0199i	0.1557 + 0.0098i
10	0.1663 - 0.0575i	0.1662 - 0.0238i	0.1557 + 0.0073i
20	0.1662 - 0.0578i	0.1660 - 0.0239i	0.1553 + 0.0073i
25	0.1662 - 0.0578i	0.1660 - 0.0239i	0.1552 + 0.0073i

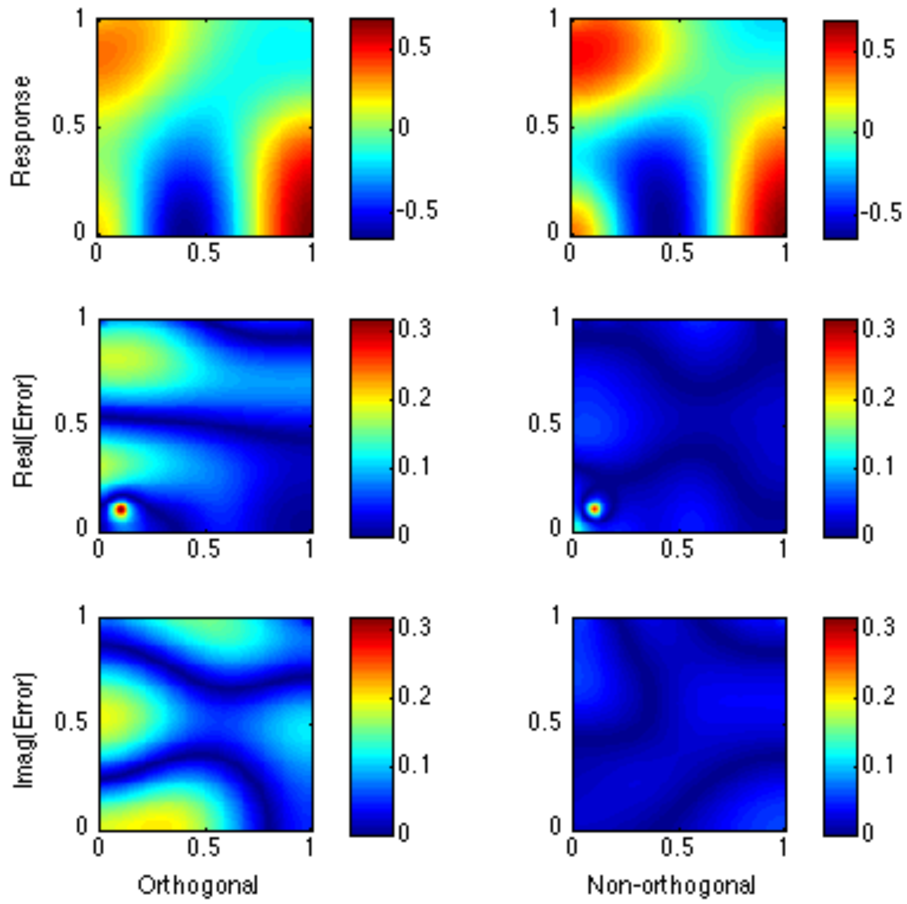

Figure 6. Response and error in real and imaginary parts of pressure field in Ω_1 using the first 25 modes compared to FEA: orthogonal approach (left) versus matrix approach (right)

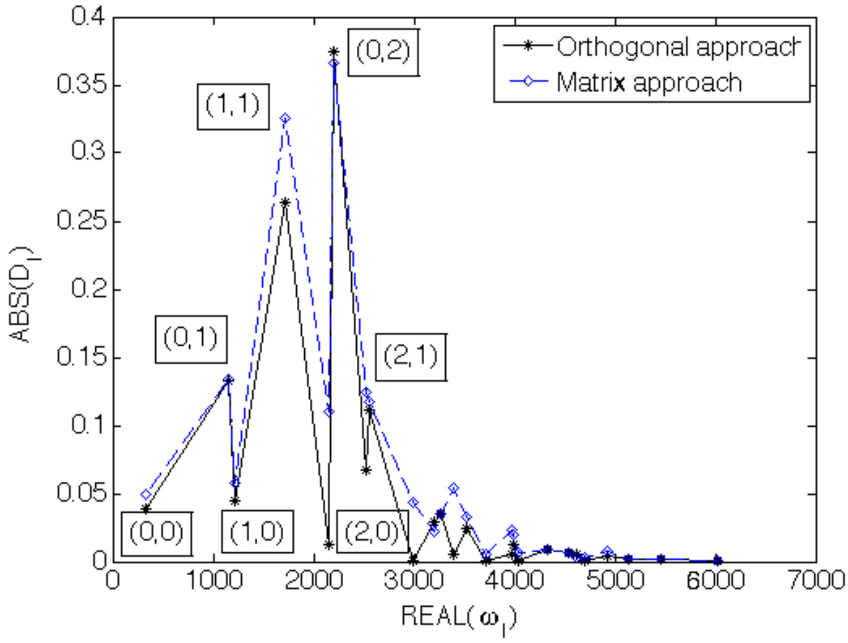

Figure 7. Contribution of modes to forced response at $\omega = 2000$ rad/s. Modes are labelled (m, n) where m is the depth mode number and n is the width mode number

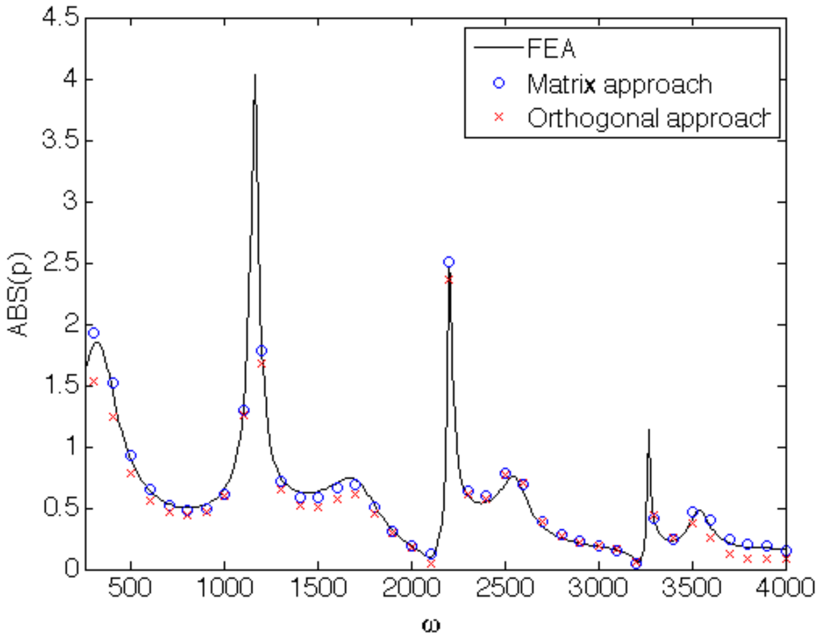

Figure 8. Comparison of the amplitude of the predicted response over a range of real excitation frequencies ω at the receiver position R1

The forced response at R1 using both approaches is generally shown to match well to FEA over a wide frequency range, with the matrix approach improving the agreement (Figure 8). Differences between the two approaches are most noticeable when visualised over the domain (Figure 6), indicating that the approximation of the cavity as orthogonal may be inadequate, especially if the continuation of the field into the exterior domain is of interest.

Conclusions

In this paper, a baffled open cavity was used as a prototype problem to investigate the completeness and orthogonality of the modes for calculating the forced response, with application to parallel noise barriers. Upon using a matrix equation to solve for the coefficients of a modal sum, the solution across the domain matched well with the FEA reference solution which suggests that completeness of the discrete modes is satisfied inside the cavity. The lack of orthogonality of the modes was shown to have a significant effect and should be included in the calculation. In this manner, the forced response of a noise source placed inside the cavity can be calculated in a computationally efficient way with a reasonable solution obtained using only a few modes, with the continuation of the weighted mode shapes yielding a prediction for the received levels in the infinite space. An interesting consideration is the case of when the noise source is instead placed above the cavity, which would require an additional term due to the continuous spectrum that represents the prompt response of the source and its reflection off the baffle. The response of the domain external to the cavity and the effect of different geometries is the subject of ongoing investigation.

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Appendix: Eigensolution for open cavity

The matrix $[\Delta]$ is defined with elements

$$\Delta_{mn} = \frac{\epsilon_m}{2}\cos\left(\sqrt{\left(\frac{\omega L}{c}\right)^2 - (n-1)^2\,\pi^2}\,\frac{D}{L}\right)\delta_{mn} \\ - \frac{i}{2}\sqrt{\left(\frac{\omega L}{c}\right)^2 - (n-1)^2\,\pi^2}\,\sin\left(\sqrt{\left(\frac{\omega L}{c}\right)^2 - \,(n-1)^2\,\pi^2}\,\frac{D}{L}\right) I_{mn}\,, \tag{11}$$

where I_{mn} defined as

$$I_{mn} = \int_0^1\int_0^1 \cos(n-1)\,\pi\rho\cos(m-1)\,\pi\eta\; H_0^{(1)}\left(\frac{\omega L}{c}|\rho-\eta|\right) d\rho\, d\eta\,, \tag{12}$$

and may be efficiently calculated using

$$I_{11} = \int_0^1 (1-y)H_0^{(1)}\left(\frac{\omega L}{c}y\right)dy\,, \tag{13}$$

$$\underset{m>1}{I_{mm}} = \int_0^1 (1-y)\cos(m-1)\pi y - \frac{1}{(m-1)\pi}\sin(m-1)\,\pi y\,)\; H_0^{(1)}\left(\frac{\omega L}{c}y\right)dy, \tag{14}$$

$$\underset{m\neq n}{I_{mn}} = \frac{1+(-1)^{m+n}}{2\pi}\left(\frac{2m-2}{(n-m)(n+m-2)}\int_0^1 (1-y)H_0^{(1)}\left(\frac{\omega L}{c}y\right)dy \right. \\ \left. - \frac{2m-2}{(n-m)(n+m-2)}\int_0^1 (1-y)H_0^{(1)}\left(\frac{\omega L}{c}y\right)dy\right)\,. \tag{15}$$

We define $H_0^{(1)}$ as the zeroth order Hankel function of the first kind, δ_{mn} is the Kronecker delta, and the elements of the vector $\{b\}$ as

$$b_m = \frac{j\omega Q}{2\pi}\int_0^1\int_{-\infty}^{\infty}\cos(m-1)\pi y\,\frac{e^{\frac{jk(y-x_0)}{L}}}{\sqrt{k^2-\left(\frac{\omega L}{c}\right)^2}}\,e^{y_0\sqrt{k^2-\frac{\omega L}{c}}}dk\,dy \tag{16}$$

with

$$\epsilon_m = \begin{cases} 2, & m=1, \\ 1, & m>1. \end{cases} \tag{17}$$